

Seminar Notes: First Properties and Equivalence in Algebraic Stacks

Zhengze Xin

March 20, 2023

Disclaimer

This is the notes for a seminar talk on Algebraic Stacks following Jarod Alper's ongoing notes. We mainly discuss sections 3.3 and 3.4 on the properties of morphisms of algebraic stacks and equivalence relations. A few typos on Jarod's notes are fixed here.

All the mistakes in this notes are mine own.

§3.3. First Properties

In this sections we define various properties of morphisms of algebraic stacks, and the necessary notions used in the definitions.

We work over big étale site $(Sch/S)_{\acute{e}t}$, where S is a base scheme.

Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks (over S). It's convenient to introduce the following terminology (following Olsson).

Definition 1.1. A *chart for f* is the commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{g} & \mathcal{X} \times_{\mathcal{Y}} V & \longrightarrow & V \\ & \searrow & \downarrow & \square & \downarrow p \\ & & \mathcal{X} & \longrightarrow & \mathcal{Y} \end{array}$$

where U and V are algebraic spaces, g and p are smooth and surjective. If we further require that X and Y are schemes, then we call this a *chart for f by schemes*.

We first discuss the properties of *stable under base change or composition*. Clearly, *representable by schemes* morphisms are stable under base change and composition, the notions of *representable* morphisms are stable under base change.

Theorem 1.2. *Representable morphisms are stable under composition.*

Proof. We first prove a lemma:

Lemma 1.3. *Let $f : X \rightarrow Y$ be a morphism of sheaves over $(\mathcal{S}ch/S)_{\text{ét}}$. If Y is an algebraic space, then X is an algebraic space.*

Proof of the lemma. We consider the chart of f by schemes as in the Def 1.1, where we take V and U to be the étale presentation of X and Y respectively. By base change the projection $X \times_Y V \rightarrow X$ is also étale, surjective and representable by schemes. So the composition $U \rightarrow X \times_Y V \rightarrow X$ is also étale, surjective and representable by schemes. Thus U is an étale presentation of X and X is an algebraic space.

Back to the proof of the theorem. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$ be the morphisms of algebraic stacks, which are representable. Then we have the following cartesian diagrams:

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} (V \times_{\mathcal{Z}} \mathcal{Y}) & \longrightarrow & \mathcal{X} \\ \downarrow f' & & \downarrow f \\ V \times_{\mathcal{Z}} \mathcal{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow g \\ V & \longrightarrow & \mathcal{Z} \end{array}$$

where V is an arbitrary scheme and $V \times_{\mathcal{Z}} \mathcal{Y}$ is an algebraic space since g is representable. By base change, f' is also representable. So by the lemma, $\mathcal{X} \times_{\mathcal{Y}} (V \times_{\mathcal{Z}} \mathcal{Y})$ is also an algebraic space. Therefore $g \circ f$ is representable. $\square$

Let $\mathcal{P}$ be a property of morphism of schemes which is stable under base change and composition. Next we define the property $\mathcal{P}$ for morphisms of algebraic (or DM) stacks.

Definition 1.4. Assume $\mathcal{P}$ is smooth (resp. étale) local on the source-and-target. We say that a representable morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of algebraic (resp. DM) stacks has property $\mathcal{P}$ if for all charts for f as in Def 1.1, the composition $U \rightarrow \mathcal{X} \times_{\mathcal{Y}} V \rightarrow V$ (as a morphism of algebraic spaces) has property $\mathcal{P}$.

Example 1.5. We can take $\mathcal{P}$ to be the property of being *smooth, locally of finite presentation or surjective*, which are smooth local on the source-and-target.

Remark. It worths mentioning that being *local on the source-and-target* is not the same as being *local on the source* and *local on the target*, because of the existence of some "bad" points on the target, locally around which $\mathcal{P}$ may not be satisfied. For a detailed discuss see [Stacks.04QW]. When the morphism is surjective (which is the case as in Jarod's notes), then the two notions have no difference, as pointed out by Junyao and Myeonhu at the ending of the talk.

Proposition 1.6. *Consider a cartesian diagram of algebraic stacks,*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{Y}' \\ \downarrow & \square & \downarrow p \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where p is smooth and surjective. Then f' is representable if and only if f is representable.

Remark. This is Prop 3.3.3 on Jarod's notes. Jarod's proof has a flaw, that **we don't need (and I think we are not able) to reduce to the case where $\mathcal{X}'$ is also a scheme.** Assuming $\mathcal{X}'$ is an algebraic space and $\mathcal{Y}', \mathcal{Y}$ are schemes, is enough for our purpose.

Meanwhile, there's a stronger version of this statement on Stacks Project ([Stacks.04ZP]), obtained following the same idea (pass to the étale neighborhoods to find sections).

Proof. The "if" direction is clear. Next we prove "only if".

First we reduce to the case where $\mathcal{X}'$ is an algebraic space and $\mathcal{Y}', \mathcal{Y}$ are schemes. Consider the following commutative cube:

$$\begin{array}{ccccc} & & \mathcal{X}' \times_{\mathcal{Y}'} U & \xrightarrow{\quad} & U \\ & \swarrow & \downarrow \text{dashed} & \swarrow & \downarrow \\ \mathcal{X}' & \xrightarrow{\quad} & \mathcal{Y}' & & \\ \downarrow & & \downarrow & & \downarrow \\ & \swarrow & \mathcal{X} \times_{\mathcal{Y}} Y & \xrightarrow{\quad} & Y \\ \mathcal{X} & \xrightarrow{\quad} & \mathcal{Y} & & \end{array}$$

where Y is an arbitrary scheme, U is a smooth presentation of $\mathcal{Y}' \times_{\mathcal{Y}} Y$. To prove that f is representable, it suffices to prove that $\mathcal{X} \times_{\mathcal{Y}} Y$ is an algebraic space. Notice that in this commutative cube, by base change, $\mathcal{X}' \times_{\mathcal{Y}'} U \rightarrow U$ is representable, and $U \rightarrow Y$ is smooth and surjective. So in our original diagram, we can just assume $\mathcal{Y}, \mathcal{Y}'$ are schemes, and $\mathcal{X}'$ is an algebraic space, and we need to prove $\mathcal{X}$ is an algebraic space. It suffices to prove that every object in $\mathcal{X}$, its stabilizer is trivial.

The idea is that, for any object in $\mathcal{X}$, we can associate it with an object in $\mathcal{Y}'$, they have the same image in $\mathcal{Y}$, thus we can lift this object to $\mathcal{X}'$, in which the stabilizer is trivial.

To do this, fix a base scheme T and consider the category $\mathcal{X}(T)$ fibered over T , an object a in $\mathcal{X}(T)$ and an automorphism $\alpha \in \underline{\text{Aut}}_T(a)$ of a . By 2-Yoneda lemma this can be identified as a 2-morphism of algebraic stacks $a : T \rightarrow \mathcal{X}$.

We have the following lemma:

Lemma 1.7. *Let $f : X \rightarrow Y$ be a morphism of schemes smooth at $x \in X$, then there exists an étale neighborhood $Y' \rightarrow Y$ of $f(x)$ such that $X \times_Y Y' \rightarrow Y'$ has a section.*

Using this lemma, we get the following commutative diagram:

$$\begin{array}{ccccccc}
 & & T' = \mathcal{X}' \times_{\mathcal{X}} T & \longrightarrow & \mathcal{X}' & \longrightarrow & \mathcal{Y}' \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow \\
 \tilde{T} & \xrightarrow{g} & T & \xrightarrow{a} & \mathcal{X} & \xrightarrow{f} & \mathcal{Y}
 \end{array}$$

where by base change $T' \rightarrow T$ is smooth surjective, and $\tilde{T} \rightarrow T$ is an étale cover which admits a lifting $\tilde{T} \rightarrow T'$ by the lemma above.

Then we get the pullback $g^*\alpha \in \underline{Aut}_T(g^*a)$. Since $\underline{Aut}_T$ is a sheaf on the big étale site $(Sch/T)_{\acute{e}t}$, it suffices to prove that $g^*\alpha$ is identity.

Now the composition $\tilde{T} \rightarrow T' \rightarrow \mathcal{X}' \rightarrow \mathcal{Y}'$ gives an object b in $\mathcal{Y}'(\tilde{T})$, which has the same image with $g^*a \in \mathcal{X}(T)$ in $\mathcal{Y}(T)$. Thus we get an object (g^*a, b) in $\mathcal{X}'(T)$, which has trivial stabilizer since $\mathcal{X}'$ is an algebraic space. Thus $(g^*\alpha, Id_b) \in \underline{Aut}_T((g^*a, b))$ is trivial, and so is $g^*\alpha$. $\square$

By effective descent, the proposition above also holds for

$$\mathcal{P} = \{\text{isomorphism, open immersion, (locally)closed immersion, affine, quasi-affine, } \dots\}.$$

Next we discuss the separation properties. Here, a question (proposed by Junyao Peng) is, **how can we discuss quasi-compactness before we introduce the topological space of algebraic stacks?**

The fact that Jarod used implicitly is that, the diagonal of the morphism of algebraic stacks is representable. In this case there's no difficulty to talk about the notions of quasi-compact and quasi-separated, since they are fpqc-local and we can proceed using the *chart of f* as in Def 1.4. In general, to define the quasi-compactness for non-representable morphisms, we will need the topological spaces of algebraic stacks. Later it's easy to show that those two definitions coincide.

Definition 1.8. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of algebraic stacks. Notice that its diagonal $\Delta_f : \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

1. We say f has affine (resp. quasi-affine, resp. separated) if the diagonal has the corresponding property. If $\mathcal{Y} = \text{Spec } \mathbb{Z}$, then we say $\mathcal{X}$ has affine (resp. quasi-affine, resp. separated) diagonal.
2. We say f is quasi-separated if its diagonal Δ_f is qcqs.
3. We say f is separated if Δ_f is proper.

Remark. At the first glimpse, it might be weird that we require Δ_f is also quasi-separated in (2). For schemes and algebraic spaces this is automatic. But for algebraic stacks this may be wrong. There exists group algebraic space which is quasi-compact but not quasi-separated, say $\mathbb{A}_k^1/\mathbb{Z}$, and the diagonal of its classifying stack BG/k is quasi-compact but not quasi-separated. There are more examples (say, [\[Stacks.06E7\]](#), [Stacks.06E9](#)) that the group algebraic space (over a base) can be non-separated or non-quasi-separated.

Proposition 1.9. *Let S be an affine scheme and $G \rightarrow S$ be a smooth affine group scheme acting on an (separated) algebraic space U over S . If U has affine diagonal, so does $[U/G]$.*

Remark. I don't know how to deduce this without assuming $U \rightarrow S$ is separated, which is necessary to apply cancellation law.

Proof. Consider the cartesian diagram:

$$\begin{array}{ccc} G \times_S U & \longrightarrow & U \times_S U \\ \downarrow & \square & \downarrow \\ [U/G] & \longrightarrow & [U/G] \times_S [U/G] \end{array}$$

where the top arrow sends (g, u) to (gu, u) .

Since G is affine, the composition $G \times_S U \rightarrow U \times_S U \xrightarrow{p_2} U$ is affine. Since $U \rightarrow S$ is separated, by the cancellation law, $G \times_S U \rightarrow U \times_S U$ is also affine. By descent the diagonal is also affine. □

To define quasi-compactness in general, we first introduce the following notions:

Definition 1.10. (Topological space of an algebraic stack) Let $\mathcal{X}$ be an algebraic stack. We can define the topological space of $\mathcal{X}$ as the set $|\mathcal{X}|$ consisting of field-valued morphism $x : \text{Spec}(K) \rightarrow \mathcal{X}$. x_1, x_2 in $|\mathcal{X}|$ are identified if they fit into the following commutative diagram:

$$\begin{array}{ccc} \text{Spec}(K) & \longrightarrow & \text{Spec}(K_1) \\ \downarrow & & \downarrow x_1 \\ \text{Spec}(K_2) & \xrightarrow{x_2} & \mathcal{X} \end{array} \quad .$$

We say $U \subset |\mathcal{X}|$ is open if there exists an open substack $\mathcal{U} \subset \mathcal{X}$ such that $U = |\mathcal{U}|$.

Definition 1.11. We say an algebraic stack $\mathcal{X}$ is quasi-compact, connected or irreducible if $|\mathcal{X}|$ is. We say $\mathcal{X}$ is noetherian if it's locally noetherian, quasi-compact and quasi-separated.

§3.4. Equivalence relations and the Luna slice theorem

Definition 1.12. An étale (resp. smooth) groupoid of schemes is the data (R, U, s, t, c) where U, R are schemes, $s, t : R \rightarrow U$ are étale (resp. smooth) morphisms called source and target respectively, and $c : R \times_{s,U,t} R \rightarrow R$ called composition, satisfy the following commutative diagrams:

$$\begin{array}{ccc}
 R \times_U R \times_U R & \xrightarrow{c \times id} & R \times_U R \\
 \downarrow id \times c & & \downarrow c \\
 R \times_U R & \xrightarrow{c} & R
 \end{array}$$

1. (associativity)

2. (identity) ...

3. (inverse) ...

When R, U are algebraic spaces and the morphisms and morphisms of algebraic spaces, we say it's an étale (resp. smooth) groupoid of algebraic spaces. If moreover $(s, t) : R \rightarrow U \times U$ is a monomorphism, then we say (R, U, s, t, c) is an étale (resp. smooth) equivalence relation.

Definition 1.13. (Quotient stack of a smooth groupoid) Let (R, U, s, t, c) be a smooth groupoid of algebraic space. We define $[U/R]^{pre}$ to be the prestack whose objects are morphisms $T \rightarrow U$ where T is a scheme. And a morphism $(S \xrightarrow{a} U) \rightarrow (T \xrightarrow{b} U)$ is the data of morphism of schemes $f : S \rightarrow T$ and an element $r \in R(S)$ such that $s(r) = a$ and $t(r) = f \circ b$.

Define $[U/R]$ to be the stackification of $[U/R]^{pre}$ in the big étale site $\mathcal{S}ch_{\acute{e}t}$.

Proposition 1.14. 1. If (R, U, s, t, c) is an étale (resp. smooth) groupoid of algebraic space, then $[U/R]$ is a DM (resp. algebraic) stack with $U \rightarrow [U/R]$ its étale (resp. smooth) presentation.

2. If (R, U, s, t, c) is an étale (resp. smooth) groupoid of schemes, then $[U/R]$ is an algebraic space with $U \rightarrow [U/R]$ its étale presentation.

Proof. We only prove for the étale case.

We need to show that $U \rightarrow \mathcal{X} = [U/R]$ is representable, surjective and étale.

First, recall that the stackification $[U/G](S)$ consists of the data $\{(S_i \rightarrow S), \{a_i\}, \{a_{ij}\}\}$ where $(S_i \rightarrow S)$ is a covering and $a_i \in [U/G]^{pre}(S)$ are compatible w.r.t the cocycle conditions. By definition of the stackification, there exists $T' \rightarrow T$ which is étale, such that we have the commutative diagram:

$$\begin{array}{ccc}
 T' & \longrightarrow & U \\
 \downarrow & & \downarrow \\
 T & \longrightarrow & \mathcal{X}
 \end{array}$$

where $T' \rightarrow U$ is obtained using the cocycle conditions. Furthermore, this fits into a commutative cube:

$$\begin{array}{ccccc}
 & & U_{T'} & \xrightarrow{\quad} & T' \\
 & \swarrow & \downarrow & \swarrow & \downarrow \\
 R & \xrightarrow{\quad} & U & & \\
 \downarrow & & \downarrow U_T & \xrightarrow{\quad} & T \\
 & \swarrow & \downarrow & \swarrow & \downarrow \\
 U & \xrightarrow{\quad} & \mathcal{X} & &
 \end{array}$$

By base change, $U_{T'} \rightarrow U_T$ is étale, surjective and representable by schemes, thus U_T is an algebraic space, and so $U \rightarrow \mathcal{X}$ is representable. Again by base change, it's also étale and surjective. Thus it's a étale presentation of $[U/R]$. □